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A hydrodynamic interpretation of a problem in the theory of the dimensional electrochemical machining of metals☆

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ABSTRACT

A method of determining the form of the anode-blank boundary for a specified form of the cathode tool in plane problems of the theory of the dimensional electrochemical machining of metals is proposed. Within the assumptions made, the anode-blank boundary is divided into a working zone, in which solution of the metal occurs, and a region next to it in which the machining ceases. The initial problem is reduced to a problem of plane-parallel potential flow of an ideal liquid with non-linear conditions on its surface. The point which separates these two regions of the anode boundary corresponds to the point where the jet separates from the solid boundary. The Brillouin-Villat smooth separation condition is specified when compiling the closed system of equations at the point where the jet separates.

The dimensional electrochemical machining of metals is based on the principle of the local solution of the anode – the blank being machined – in a flowing electrolyte. An electrode with a specified geometrical shape of the surface acts as the cathode – the machining tool. A detailed description of the process and of the technology of dimensional electrochemical machining can be found in Refs 1–4.

To obtain high accuracy in copying the shape and size of the cathode on the blank being machined with a specified margin on the machining it is necessary to localize the electrochemical solution of the metal in the zone to be machined. Outside the limits of this zone the solution of the metal must be slowed down sharply until it completely ceases. On the basis of an analysis of the electroprocesses for different electrolytes it has been shown[4] that localization of the solution of the metal depends considerably on the composition of the electrolyte, the properties of the metal and the conditions under which the machining is carried out. For small values of the current density a value of the current yield η for reactions of the anode solution of the metal in sodium nitrate and chlorate solutions are practically zero. As a result of anode activation of the metal[4] due to the action of anions of the salts, at a certain critical current density, η begins to increase as the current density increases. Then the main solution of the metal is concentrated on those parts of the blank being machined where the interelectrode spacing is least, while the rate of solution is a maximum.

When the necessary conditions are satisfied, after prolonged machining, the surface takes a definite shape, that is constant in time, which is said to be stable or stationary.[4] In the stable mode, the shape of the surface being machined in a mobile system of coordinates, connected with the cathode, does not change, i.e., the anode surface is shifted together with the cathode at a constant rate.

The problems involved in establishing the relation between the shapes of the cathode-tool and the blank obtained in electrochemical machining are called electrochemical machining problems. Formulations of the problems in the model of the ideal process and methods for solving them are well known.[4] In the case of a two-dimensional electric field, if the value of the current yield η is a constant quantity, the hodograph of the electric field strength at the anode is part of a circumference. This is the basis of the hodograph method.[5] The two-dimensional problem of determining the stationary boundaries of the anode, taking into account the dependence of η on the current density, was solved in Ref. 6 using the boundary element method.

Below, using a model of the ideal process,[4] we find a numerical-analytical solution of the plane problem of the theory of electrochemical machining, involving the determination of the stable form of the surface of the blank when it is treated with a trihedral cathode of symmetrical shape, taking into account the localizing properties of sodium nitrate electrolyte. Unlike the problems considered previously,[4–6] when formulating and solving this problem we take into account the transition from the zone of intensive solution of the metal into the region where the anode current is very small and no solution of the metal occurs.

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1. The model of the process

The electrochemical machining process is described, in the general case, by the system of equations of motion of a viscous multiphase electrically conducting liquid, Maxwell's equations, the energy equation, the convective diffusion equation and the equation of state of a gas and also the dependence of the thermal parameters on the temperature, pressure and state of the medium. The aim of mathematical modelling of the process is to determine the shape of the surface of one of the electrodes for a specified shape of the other. We will use as a first approximation a model of the ideal process; the basic assumptions and a detailed justification of them were given previously in Ref. 4.

According to the model of the ideal process the electric field with a potential u in the interelectrode space can be described by Laplace's equation

$$\nabla^2 u = 0 \tag{1.1}$$

The potential $u_a = U - E_a$ on the anode surface is determined by the difference between the specified fixed value of the voltage U between the electrodes and the jump in the electrode potential E_a of the anode. The potential $u_c = -E_c$ on the cathode working surface is identical with the value of the jump in the cathode electrode potential. It is assumed in the model of the ideal process that E_a and E_c are constant potential jumps, averaged over the surface of the electrodes.[4]

Henceforth we will consider the plane problem. We introduce a system of Cartesian coordinates x_1 and y_1, connected with the cathode. We will assume that the cathode moves in the direction of the ordinate axis.

Starting from Eq. (1.1), we can assume that the function $u(x_1, y_1)$ is the imaginary part of the analytic function $f(z_1) = v(x_1, y_1) + iu(x_1, y_1)$ of the complex variable $z_1 = x_1 + iy_1$. The function $f(z_1)$ is the complex potential of the electrostatic field, and its real part $v(x_1, y_1)$ is a function of the current. The level lines $u(x_1, y_1) = \text{const}$ are equipotential field lines, while the lines $v(x_1, y_1) = \text{const}$ are lines of force. The field-strength vector $\mathbf{E}(x_1, y_1) = -\text{grad}\, u(x_1, y_1)$ is expressed in terms of the complex potential, and consequently, all the quantities characterizing the field also.[7]

The current density distribution on the steady anode boundary is given by the equation[4]

$$\eta(i_a) i_a = \varepsilon^{-1} \rho V_c \cos\theta \tag{1.2}$$

where $i_a = \kappa \partial u / \partial n_a$ is the anode current density, κ is the electrical conductivity of the medium, ε is the electrochemical equivalent of the metal, ρ is the density of the anode material and θ is the angle between the vector V_c of the velocity of feed of the cathode and the vector n_a of the outward normal at a given point of the anode boundary. In condition (1.2) it is assumed that the angle η is a function of i_a.

Graphs of the current yield against the current density when machining 5KhNM steel in solutions of sodium nitrate and chlorate of different concentrations were presented in Ref. 8. For these electrolytes we can represent the analytical relation $\eta(i_a)$ in the form[6]

$$\eta(i_a) = \begin{cases} 0, & i_a \le i_{\text{cr}} \\ a_0 + a_1/i_a, & i_a > i_{\text{cr}} \end{cases} \tag{1.3}$$

Here a_0, a_1 and i_{cr} are constant quantities. In Fig. 1 we show a graph of $\eta(i_a)$ when machining 5KhNM steel in a solution of sodium nitrate with a concentration of 15%. The continuous curve represents the graph of the function (1.3) for this special case while the points represent experimental results.[8]

Using relations (1.2) and (1.3), we obtain the boundary condition on the steady anode boundary

$$\kappa \frac{\partial u}{\partial n_a} = -\frac{a_1}{a_0} + \frac{\rho V_c}{a_0 \varepsilon} \cos\theta \tag{1.4}$$

We will now introduce the characteristic current density i_0, the characteristic length H (Ref. 5) and the dimensionless variables

$$i_0 = \rho V_c/\varepsilon, \quad H = \kappa(u_a - u_c)/i_0, \quad x = x_1/H, \quad y = y_1/H, \quad n = n_a/H$$

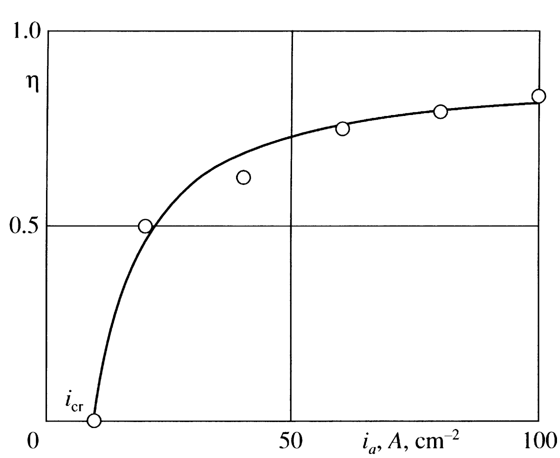

Fig. 1.

and we will change to the dimensionless complex potential

$$W(z) = \varphi(x, y) + i\psi(x, y); \quad z = x + iy$$

Using the transformation[5]

$$W(z) = (f(z) - iu_c)/(u_a - u_c)$$

Then the function ψ satisfies Laplace's equation in the interelectrode space with the following conditions on the electrode boundaries

$$\psi_a = 1, \quad \psi_c = 0 \tag{1.5}$$

and on the unknown anode boundary

$$\partial\psi/\partial n = a + b\cos\theta; \quad a = -a_1/a_0 i_0, \quad b = 1/a_0 \tag{1.6}$$

In special cases the cathode may have lines of symmetry or parts of the boundaries of the dielectric coatings, which are deposited on the non-working surfaces of the electrode-tool. The following condition is satisfied on the lines of symmetry and on the boundaries of the dielectric coatings

$$\partial\psi/\partial n = 0 \tag{1.7}$$

According to the hydrodynamic analogy[9] a plane potential electric field is modelled by a fictitious plane-parallel potential flow of an ideal incompressible fluid. The stream function of the fictitious flow corresponds to the electric field potential, while the velocity potential corresponds to the stream function. The hydrodynamic analogy of the electric field strength E is the velocity V of this flow; it should be noted that the vectors V and E are mutually orthogonal.[7] Then, in the case of the hydrodynamic interpretation the equality $\partial\psi/\partial n = V$ is satisfied along the streamline ψ = const, where $V = |\mathbf{V}|$. In this system of coordinates the abscissa axis is orthogonal to the direction of approach of the cathode. In this case the slope of the vector **V** to the abscissa axis at a given point of the anode boundary is identical, apart from the sign, with the angle between the direction of approach of the cathode and the vector of the outward normal at the same point. Then, by condition (1.3) on the anode boundary the velocity of the fictitious flow varies as follows:

$$V = a + b\cos\theta, \tag{1.8}$$

where θ is the argument of the velocity vector.

The problem of determining the shape of the anode boundary, in its hydrodynamic interpretation, corresponds to the problem of the flow of an ideal fluid with free surfaces.[10] The hydrodynamic analogy facilitates the formulation of the boundary-value problems and enables methods of the theory of jets of an ideal fluid[10,11] to be applied to problems of the dimensional electrochemical machining of metals.

2. Formulation of the problem and its numerical-analytical solution

A sketch of the cross section of the interelectrode gap is shown in Fig. 2. The cross-section of the cathode is an isosceles triangle with angle at the base $\alpha\pi$. In view of the symmetry of the interlectrode gap we can confine ourselves to considering its left-hand part. The polygon CDE on it corresponds to the boundary of the cathode. The axes of symmetry BC and EF are electric current lines, orthogonal to the equipotential field lines.

The electrode-tool of this scheme can be used to cut deep grooves in machine components, channels in blanks and other technological processes.

According to condition (1.3) the required anode boundary can be divided into two regions. Solution of the metal occurs in region AB. The normal derivative $\partial\psi/\partial n$ in this part satisfies condition (1.6). In the region modelled by the vertical rectilinear part AF, the anode current

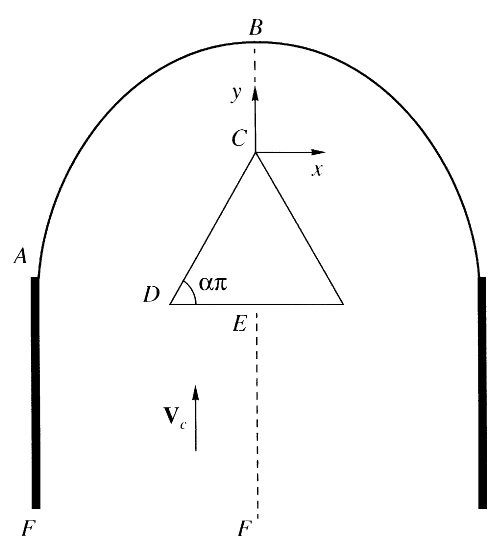

Fig. 2.

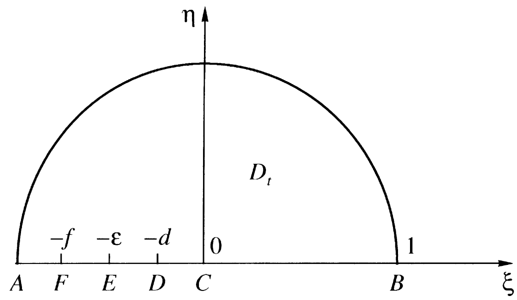

Fig. 3.

yield is practically equal to zero and no solution of the metal occurs. The current density on the part AF changes from a value i_{cr} at the point A to zero at an infinitely distant point F. The position of the point A is unknown in advance and must be obtained when solving the problem. The vector $\mathbf{V}_c$ indicates the direction of motion of the cathode. The abscissa axis is chosen to be orthogonal to the direction of motion of the cathode.

The hydrodynamic analogue is the problem of the theory of plane steady flows of an ideal incompressible fluid along a certain boundary AB with the specified velocity variation (1.8). The flow is produced by a system of sources, continuously distributed along the line EF, and sinks on the line BC.

Starting from a priori notion on the shape of the anode boundary AB, we conclude that in this section the angle θ decreases monotonically from π/2 to zero, i.e., there are no inflection points on the anode boundary AB. Along the boundary AF the velocity must fall monotonically from a constant value $V = a$ at the point A to zero at an infinitely distant point *F*. To ensure these requirements we will use the Brillouin-Villat smooth separation condition, well known in hydrodynamics.[10,11] According to this condition the curvature of the anode boundary AB at the point A is finite and coincides with the curvature of the wall AF, i.e., equal to zero for this problem.

To solve the problem we will introduce an auxiliary complex variable $t = \xi + i\eta$, which varies in the region $D_t = \{|t| \le 1, \eta \ge 0\}$ (Fig. 3), and we will seek a function $z(t)$ which conformally maps a semicircle of unit radius in the flow region corresponding to the points shown in Figs. 2 and 3.

Instead of the function $z(t)$ we can seek the Zhukovskii function[10]

$$\chi(t) = \ln\left(\frac{1}{V_0}\frac{dW}{dz}\right) = r - i\theta; \quad r = \ln\frac{V}{V_0} \tag{2.1}$$

where $V_0 = a + b$ is the value of the velocity of the fictitious flow at the point B ($t = 1$). The function $\chi(t)$ is related to the functions $W(t)$ and $z(t)$ by the equation

$$\frac{dz(t)}{dt} = \frac{\exp(-\chi(t))}{V_0}\frac{dW}{dt} \tag{2.2}$$

According to conditions (1.5) the complex potential $W(t) = \varphi(t) + i\psi(t)$ satisfies the following boundary conditions

$$\psi(t) = \begin{cases} 1, & t = \exp(i\sigma), \quad \sigma \in [0, \pi] \\ 1, & t = \xi, \quad \xi \in [-1, -f] \\ 0, & t = \xi, \quad \xi \in [-\varepsilon, 0] \end{cases}$$

It follows from condition (1.7) that the function $\varphi(t)$ takes constant values on the symmetry lines EF and BC. We will assume that

$$\varphi(t) = \begin{cases} 0, & t = \xi, \quad \xi \in [-f, -\varepsilon] \\ \varphi_0, & t = \xi, \quad \xi \in [0, 1] \end{cases}$$

The region D_w in which the complex potential varies is a rectangle with sides φ_0 and 1 (Fig. 4).

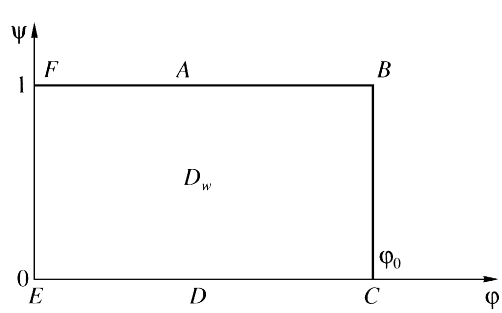

Fig. 4.

Using the method of conformal mappings, we obtain the derivative of the complex potential

$$\frac{dW}{dt} = N\frac{1+t}{\sqrt{\tau(t)}}$$

$$N = \frac{i}{I_0},\quad I_0 = \int_0^1 \frac{(1+x)dx}{\sqrt{\tau(x)}},\quad \tau(t) = t(t+\varepsilon)(1+t\varepsilon)(t+f)(1+tf) \tag{2.3}$$

The parameter φ_0, representing the current in the electrochemical cell,[3] is given by the formula

$$\varphi_0 = \frac{1}{I_0}\int_{-\varepsilon}^{0} \frac{(1+x)dx}{\sqrt{-\tau(x)}}$$

We will represent the function $\chi(t)$ in the form of a sum[10]

$$\chi(t) = \chi_*(t) + \omega(t) \tag{2.4}$$

where $\omega(t)$ is a function which is analytical in the region where the variable t changes, while the function $\chi_*(t) = r_* = i\theta_*$, $r_* = \ln(V_*/V_0)$ corresponds to the flow in the specified scheme (Fig. 2) with the condition $V_* = V_0$ on the anode boundary AB. It follows from condition (1.8) and the flow scheme shown in Fig. 2 that, on the boundary of the region D_t, the functions $\chi(t)$ and $\chi_*(t)$ satisfy the conditions

$$a + b\cos\theta(t) - V_0\exp(r(t)) = 0,\quad \mathrm{Re}\chi_*(t) = 0$$

$$t = \exp(i\sigma),\quad \sigma\in[0,\pi],\quad r(1) = 0$$

$$\mathrm{Im}\chi(\xi) = \mathrm{Im}\chi_*(\xi) = \begin{cases} -\pi/2, & \xi\in[-1,-f) \\ -\pi, & \xi\in(-f,-d) \\ -\alpha\pi, & \xi\in(-d,0) \\ 0, & \xi\in(0,1] \end{cases} \tag{2.5}$$

Using Chaplygin's method of singular points,[10] we obtain

$$\chi_*(t) = \frac{1}{2}\ln\frac{t+f}{1+tf} - (1-\alpha)\ln\frac{t+d}{1+td} - \alpha\ln t \tag{2.6}$$

where d and f are the coordinates of the sections of the points D and F in the region D_t.

Taking equality (2.4) and boundary conditions (2.5) into account, we obtain the following non-linear boundary-value problem for the function $\omega(t)$

$$a + b\cos(T+\mu) - V_0\exp(\lambda) = 0 \tag{2.7}$$

$$\mathrm{Im}\omega(\xi) = 0,\quad \xi\in[-1,1],\quad \mathrm{Re}\omega(1) = 0 \tag{2.8}$$

Here

$$T = \mathrm{Im}\chi_*(\exp(i\sigma)),\quad \mu = \mathrm{Im}\omega(\exp(i\sigma)),\quad \lambda = \mathrm{Re}\omega(\exp(i\sigma))$$

The function $\omega(t)$, which gives a solution of boundary-value problem (2.7), (2.8), can be expanded, by virtue of condition (2.8), in a power series with real coefficients

$$\omega(t) = \sum_{k=0}^{\infty} c_k t^k;\quad c_0 = -\sum_{k=1}^{\infty} c_k \tag{2.9}$$

The condition for smooth separation at the point A can be represented[10,11] in the form of an equality

$$d\theta/d\sigma = 0 \text{ when } \sigma = \pi \tag{2.10}$$

which, using formulae (2.4), (2.6) and (2.9), can be reduced to the form

$$f = \frac{2F-1}{2F+1};\quad F = \alpha + (1-\alpha)\frac{1+d}{1-d} - \sum_{k=1}^{\infty}(-1)^k c_k k \tag{2.11}$$

All the necessary geometrical characteristics of the flow can be found using the parametric representation (2.2):

$$dz = iM\left(\frac{1+t}{t+f}\right)\left(\frac{t+d}{1+td}\right)^{1-\alpha}\frac{t^{\alpha-1/2}}{\sqrt{(t+\varepsilon)(1+t\varepsilon)}}\exp\left(-\sum_{k=1}^{\infty}c_k t^k\right)dt \tag{2.12}$$

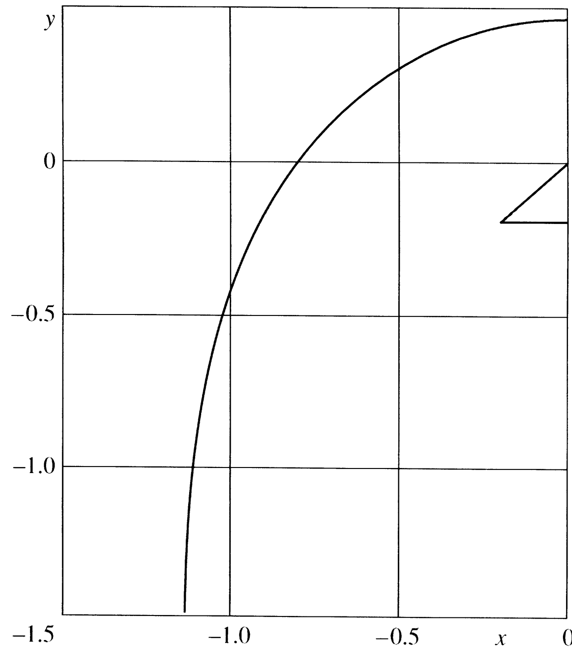

Fig. 5.

where

$$M = 1/(V_0 I_0 \exp(c_0))$$

Integrating expression (2.12) in the sections $[-\varepsilon, -d]$ and $[-d, 0]$, we obtain the length L of the section DE and the length L_1 of the section CD

$$L = L(d, f, \varepsilon), \quad L_1 = L_1(d, f, \varepsilon); \quad L = L_1 \cos\alpha\pi \tag{2.13}$$

For the numerical solution of the problem we specify the geometrical quantities L and α, the coefficients a_0 and a_1, representing the properties of the electrolyte, and the characteristic current density i_0. The expansion coefficients (2.9) are defined in such a way that condition (2.7) is satisfied on the required anode boundary. The problem is solved numerically by the collocation method, which is widely employed in hydrodynamics problems.[10] The system of equations for calculating the expansion coefficients (2.9) is solved by Newton's method together with Eqs (2.11) and (2.13), intended for determining the parameters d, ε and f.

3. The results of numerical calculations

To estimate the accuracy of the numerical results as a function of the number of collocation points N, we carried out test calculations for the following values of the specified parameters

$$L = 0.2, \quad \alpha = 0.25, \quad i_0 = 100 \text{ A/cm}^2, \quad a_0 = 0.906, \quad a_1 = -12.818$$

(the values of a_0 and a_1 correspond to a solution of sodium nitrate with a concentration of 15%, and $a = 0.141$ and $b = 1.104$ (Ref 6)). For $N = 120$ the approximate solution for this special case can be found with an accuracy of 10^{-4}; the values of the parameters d, ε and f and the coordinates of the point A are then

$$d = 0.443, \quad \varepsilon = 0.573, \quad f = 0.912, \quad x = -1.132, \quad y = -1.488$$

In Fig. 5 we show the results of a calculation of the shape of the anode boundary for this special case.

If the smooth separation condition is not used, this leads to a breakdown of the condition for a monotonic change in the velocity along the free boundary AB and the boundary AF, which does not enable the free boundary, satisfying boundary condition (1.8), to be constructed.

4. Conclusion

Using a two-dimensional mathematical model of the ideal process[4] when calculating the specific dependence of the current yield on the anode current density we have solved the problem of calculating the shape of the steady anode boundary for a specified cathode configuration. The use of the smooth separation condition[10,11] enables one to determine the anode boundary which satisfies boundary condition (1.8). This uniquely ensures the possible value of the interelectrode gap between the symmetry line and the vertical part of the surface being machined.

References

1. Moroz II, Alekseyev GA, Vodyanitskii OA, et al. *The Electrochemical Machining of Metals*. Moscow: Mashinostroyeniye; 1969.
2. Sedykin FV. *The Dimensional Electrochemical Machining of Machine Parts*. Moscow: Mashinostroyeniye; 1976.
3. Scherbak MV, Tolstaya MA, Anisimov AP, et al. *Principles of the Theory and Practice of the Electrochemical Treatment of Metals and Alloys*. Moscow: Mashinostroyeniye; 1981.
4. Davydov AD, Kozak Ye. *High-Speed Electrochemical Forming*. Moscow: Nauka; 1990.
5. Klokov VV. *Electrochemical Forming*. Kazan: Izd Kazan Univ; 1984.
6. Kotlyar LM, Minazetdinov NM. Determination of the shape of the anode taking into account the electrolyte properties in problems of the electrochemical dimensional machining of metals. *Zh Prikl Mekh Tekh Fiz* 2003;**44**(3):170–84.
7. Lavrent'ev MA, Shabat BV. *Methods of the Theory of Functions of a Complex Variable*. Moscow: Nauka; 1987.
8. Sedykin FV, Orlov BP, Matasov VF. Investigation of the anode current yield in electrochemical machining using a constant and pulsed voltage. *Tekh Mashinostroyeniye Tula* 1975;**39**:3–10.
9. Klokov VV, Kosterin AV, Nuzhin MT. The application of inverse boundary-value problems in the theory of dimensional electrochemical machining. In Proc Seminar on Boundary-Value Problems. Kazan; 1972, 9, 132–40.
10. Gurevich MI. *The Theory of Jets in on Ideal Fluid*. Oxford: Pergamon; 1996.
11. Birkhoff G, Zarantonello E. *Jets, Wakes and Cavities*. New York: Academic Press; 1957.

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